

Algebraic Number Theory

(PARI-GP version 2.9.0)

Binary Quadratic Forms

create $ax^2 + bxy + cy^2$ (distance d) `Qfb(a,b,c,{d})`
reduce x ($s = \sqrt{D}$, $l = \lfloor s \rfloor$) `qfbred(x,{flag},{D},{l},{s})`
return $[y, g]$, $g \in \text{SL}_2(\mathbf{Z})$, $y = g \cdot x$ reduced `qfbreds12(x)`
composition of forms $x*y$ or `qfbnucomp(x,y,l)`
 n -th power of form x^n or `qfbnupow(x,n)`
composition without reduction `qfbcomprow(x,y)`
 n -th power without reduction `qfbpowrow(x,n)`
prime form of disc. x above prime p `qfbprimeform(x,p)`
class number of disc. x `qfbclassno(x)`
Hurwitz class number of disc. x `qfbhclassno(x)`
Solve $Q(x,y) = p$ in integers, p prime `qfbsolve(Q,p)`

Quadratic Fields

quadratic number $\omega = \sqrt{x}$ or $(1 + \sqrt{x})/2$ `quadgen(x)`
minimal polynomial of ω `quadpoly(x)`
discriminant of $\mathbf{Q}(\sqrt{D})$ `quaddisc(x)`
regulator of real quadratic field `quadregulator(x)`
fundamental unit in real $\mathbf{Q}(x)$ `quadunit(x)`
class group of $\mathbf{Q}(\sqrt{D})$ `quadclassunit(D,{flag},{t})`
Hilbert class field of $\mathbf{Q}(\sqrt{D})$ `quadhilbert(D,{flag})`
... using specific class invariant ($D < 0$) `polclass(D,{inv})`
ray class field modulo f of $\mathbf{Q}(\sqrt{D})$ `quadray(D,f,{flag})`

General Number Fields: Initializations

The number field $K = \mathbf{Q}[X]/(f)$ is given by irreducible $f \in \mathbf{Q}[X]$.
A nf computes a maximal order and allows operations on elements and ideals. A bnf adds class group and units. A bnr is attached to ray class groups and class field theory. A rnf is attached to relative extensions L/K .

init number field structure nf `nfinit(f,{flag})`
known integer basis B `nfinit([f,B])`
order maximal at $vp = [p_1, \dots, p_k]$ `nfinit([f,vp])`
order maximal at all $p \leq P$ `nfinit([f,P])`
certify maximal order `nfcertify(nf)`

nf members:

a monic $F \in \mathbf{Z}[X]$ defining K $nf.pol$
number of real/complex places $nf.r1/r2/sign$
discriminant of nf $nf.disc$
 T_2 matrix $nf.t2$
complex roots of F $nf.roots$
integral basis of \mathbf{Z}_K as powers of θ $nf.zk$
different/codifferent $nf.diff$, $nf.codiff$
index $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$ $nf.index$
recompute nf using current precision $nf.newprec(nf)$
init relative rnf $L = K[Y]/(g)$ $rnfinit(nf,g)$
init bnf structure $bnfinit(f,{flag})$

bnf members:

underlying nf $bnf.nf$
classgroup $bnf.clgp$
regulator $bnf.reg$
fundamental/torsion units $bnf.fu$, $bnf.tu$
compress a bnf for storage $bnf.compress(bnf)$
recover a bnf from compressed $bnfz$ $bnfinit(bnfz)$
add S -class group and units, yield $bnfS$ $bnfsunit(bnf,S)$
init class field structure bnr $bnrinit(bnf,m,{flag})$

bnr members:

same as bnf , plus
underlying bnf $bnr.bnf$
big ideal structure $bnr.bid$
modulus $bnr.mod$
structure of $(\mathbf{Z}_K/m)^*$ $bnr.zkst$

Basic Number Field Arithmetic (nf)

Elements are `t_INT`, `t_FRAC`, `t_POL`, `t_POLMOD`, or `t_COL` (on integral basis $nf.zk$). Basic operations (prefix `nfelt`): (`nfelt`)`add`, `mul`, `pow`, `div`, `diveuc`, `mod`, `divrem`, `val`, `trace`, `norm`
express x on integer basis `nfalgtobasis(nf,x)`
express element x as a polmod `nfbasistoalg(nf,x)`
complex embeddings of `t_POLMOD` x `conjvec(x)`
reverse polmod $a = A(X) \bmod T(X)$ `modreverse(a)`
integral basis of field def. by $f = 0$ `nfbasis(f)`
field discriminant of field $f = 0$ `nfdisc(f)`
smallest poly defining $f = 0$ (slow) `polredabs(f,{flag})`
small poly defining $f = 0$ (fast) `polredbest(f,{flag})`
random Tschirnhausen transform of f `poltschirnhaus(f)`
 $\mathbf{Q}[x]/(f) \subset \mathbf{Q}[x]/(g)$? Isomorphic? `nfisincl(f,g)`, `nfisisom`
compositum of $\mathbf{Q}[X]/(f)$, $\mathbf{Q}[X]/(g)$ `polcompositum(f,g,{flag})`
compositum of $K[X]/(f)$, $K[X]/(g)$ `nfcompositum(nf,f,g,{flag})`
splitting field of K (degree divides d) `nfsplitting(nf,{d})`
subfields (of degree d) of nf `nfsubfields(nf,{d})`
 d -th degree subfield of $\mathbf{Q}(\zeta_n)$ `polsubcyclo(n,d,{v})`
roots of unity in nf `nfrootsof1(nf)`
roots of g belonging to nf `nfroots({nf},g)`
factor g in nf `nfactor(nf,g)`
factor $g \bmod$ prime pr in nf `nfactormod(nf,g,pr)`
conjugates of a root θ of nf `nfgaloisconj(nf,{flag})`
apply Galois automorphism s to x `nfgaloisapply(nf,s,x)`
quadratic Hilbert symbol (at p) `nfhilbert(nf,a,b,{p})`

Linear and algebraic relations

poly of degree $\leq k$ with root $x \in \mathbf{C}$ `algdep(x,k)`
alg. dep. with pol. coeffs for series s `seralgdep(s,x,y)`
small linear rel. on coords of vector x `linddep(x)`

Dedekind Zeta Function ζ_K , Hecke L series

$R = [c, w, h]$ in initialization means we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w$, $|\Im(s)| < h$; $R = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $R = [1/2, 0, h]$ (critical line up to height h).

ζ_K as Dirichlet series, $N(I) < b$ `dirzetak(nf,b)`
init $\zeta_K^{(k)}(s)$ for $k \leq n$ `L = lfuninit(bnf,R,{n=0})`
compute $\zeta_K(s)$ (n -th derivative) `lfun(L,s,{n=0})`
compute $\Lambda_K(s)$ (n -th derivative) `lfunlambda(L,s,{n=0})`

init $L_K^{(k)}(s, \chi)$ for $k \leq n$ `L = lfuninit([bnr,chi],R,{n=0})`
compute $L_K(s, \chi)$ (n -th derivative) `lfun(L,s,{n})`
Artin root number of K `bnrrootnumber(bnr,chi,{flag})`
 $L(1, \chi)$, for all χ trivial on H `bnrL1(bnr,{H},{flag})`

Class Groups & Units (bnf, bnr)

Class field theory data $a_1, \{a_2\}$ is usually bnr (ray class field), bnr, H (congruence subgroup) or bnr, χ (character on `bnr.clgp`). Any of these define a unique abelian extension of K .
remove GRH assumption from bnf `bnfcertify(bnf)`
expo. of ideal x on class gp `bnfisprincipal(bnf,x,{flag})`
expo. of ideal x on ray class gp `bnrisprincipal(bnr,x,{flag})`
expo. of x on fund. units `bnfisunit(bnf,x)`
as above for S -units `bnfissunit(bnfs,x)`

signs of real embeddings of $bnf.fu$ `bnfsignunit(bnf)`
narrow class group `bnfnarrow(bnf)`
Class Field Theory
ray class number for modulus m `bnrclassno(bnf,m)`
discriminant of class field `bnrdisc(a_1,{a_2})`
ray class numbers, l list of moduli `bnrclassolist(bnf,l)`
discriminants of class fields `bnrdisclist(bnf,l,{arch},{flag})`
decode output from `bnrdisclist` `bnfdecodemodule(nf,fa)`
is modulus the conductor? `bnrisconductor(a_1,{a_2})`
is class field (bnr, H) Galois over K^G `bnrisgalois(bnr,G,H)`
action of automorphism on `bnr.gen` `bnrgaloismatrix(bnr,aut)`
apply `bnrgaloismatrix` M to H `bnrgaloisapply(bnr,M,H)`
characters on `bnr.clgp` s.t. $\chi(g_i) = e(v_i)$ `bnrchar(bnr,g,{v})`
conductor of character χ `bnrconductor(bnr,chi)`
conductor of extension `bnrconductor(a_1,{a_2},{flag})`
conductor of extension $K[Y]/(g)$ `rnfconductor(bnf,g)`
Artin group of extension $K[Y]/(g)$ `rnfnormgroup(bnr,g)`
subgroups of bnr , index $\leq b$ `subgrouplist(bnr,b,{flag})`
rel. eq. for class field def'd by sub `rnfkummer(bnr,sub,{d})`
same, using Stark units (real field) `bnrstark(bnr,sub,{flag})`
is a an n -th power in K_v ? `nfislocalpower(nf,v,a,n)`
cyclic L/K satisf. local conditions `nfgrunwaldwang(nf,P,D,pl)`

Logarithmic class group

logarithmic ℓ -class group `bnflog(bnf,l)`
 $[\tilde{e}(F_v/Q_p), \tilde{f}(F_v/Q_p)]$ `bnflogef(bnf,pr)`
 $\exp \deg_F(A)$ `bnflogdegree(bnf,A,l)`
is ℓ -extension L/K locally cyclotomic `rnfislocalcyclo(rmf)`

Ideals:

elements, primes, or matrix of generators in HNF
is id an ideal in nf ? `nfisideal(nf,id)`
is x principal in bnf ? `bnfisprincipal(bnf,x)`
give $[a, b]$, s.t. $a\mathbf{Z}_K + b\mathbf{Z}_K = x$ `idealtwoelt(nf,x,{a})`
put ideal a ($a\mathbf{Z}_K + b\mathbf{Z}_K$) in HNF form `idealhnf(nf,a,{b})`
norm of ideal x `idealnrm(nf,x)`
minimum of ideal x (direction v) `idealmin(nf,x,v)`
LLL-reduce the ideal x (direction v) `idealred(nf,x,{v})`

Ideal Operations

add ideals x and y `idealadd(nf,x,y)`
multiply ideals x and y `idealmul(nf,x,y,{flag})`
intersection of ideals x and y `idealintersect(nf,x,y,{flag})`
 n -th power of ideal x `idealpow(nf,x,n,{flag})`
inverse of ideal x `idealinu(nf,x)`
divide ideal x by y `idealdiv(nf,x,y,{flag})`
Find $(a, b) \in x \times y$, $a + b = 1$ `idealaddtoone(nf,x,{y})`
coprime integral A, B such that $x = A/B$ `idealnumden(nf,x)`

Primes and Multiplicative Structure

factor ideal x in \mathbf{Z}_K `idealfactor(nf,x)`
expand ideal factorization in K `idealfactorback(nf,f,{e})`
expand elt factorisation in K `nffactorback(nf,f,{e})`
decomposition of prime p in \mathbf{Z}_K `idealprimedec(nf,p)`
valuation of x at prime ideal pr `idealval(nf,x,pr)`
weak approximation theorem in nf `idealchinese(nf,x,y)`
 $a \in K$, s.t. $v_p(a) = v_p(x)$ if $v_p(x) \neq 0$ `idealappr(nf,x)`
 $a \in K$ such that $(a \cdot x, y) = 1$ `idealcoprime(nf,x,y)`
give bid = structure of $(\mathbf{Z}_K/id)^*$ `idealstar(nf,id,{flag})`
structure of $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$ `idealprincipalunits(nf,pr,k)`
discrete log of x in $(\mathbf{Z}_K/bid)^*$ `ideallog(nf,x,bid)`

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idealstar of all ideals of norm $\leq b$ **ideallist**($nf, b, \{flag\}$)
 add Archimedean places **ideallistarch**($nf, b, \{ar\}, \{flag\}$)
 init **modpr** structure **nfmodprinit**(nf, pr)
 project t to \mathbf{Z}_K/pr **nfmodpr**($nf, t, modpr$)
 lift from \mathbf{Z}_K/pr **nfmodprlift**($nf, t, modpr$)

Galois theory over \mathbf{Q}

Galois group of field $\mathbf{Q}[x]/(f)$ **polgalois**(f)
 initializes a Galois group structure G **galoisinit**($pol, \{den\}$)
 action of p in **nfgaloisconj** form **galoispermopol**($G, \{p\}$)
 identify as abstract group **galoisidentify**(G)
 export a group for GAP/MAGMA **galoisexport**($G, \{flag\}$)
 subgroups of the Galois group G **galoissubgroups**(G)
 is subgroup H normal? **galoisisnormal**(G, H)
 subfields from subgroups **galoissubfields**($G, \{flag\}, \{v\}$)
 fixed field **galoisfixedfield**($G, perm, \{flag\}, \{v\}$)
 Frobenius at maximal ideal P **idealfrobenius**(nf, G, P)
 ramification groups at P **idealramgroups**(nf, G, P)
 is G abelian? **galoisisabelian**($G, \{flag\}$)
 abelian number fields/ \mathbf{Q} **galoissubcyclo**($\mathbf{N}, \mathbf{H}, \{flag\}, \{v\}$)
 query the **galpol** package **galoisgetpol**($a, b, \{s\}$)

Relative Number Fields (rnf)

Extension L/K is defined by $T \in K[x]$.
 absolute equation of L **rnfequation**($nf, T, \{flag\}$)
 is L/K abelian? **rnfisabelian**(nf, T)
 relative **nfalttobasis** **rnfalttobasis**(rnf, x)
 relative **nfbasistoalg** **rnfbasistoalg**(rnf, x)
 relative **idealhnf** **rnfidealhnf**(rnf, x)
 relative **idealmul** **rnfidealmul**(rnf, x, y)
 relative **idealtwoelt** **rnfidealtwoelt**(rnf, x)

Lifts and Push-downs

absolute \rightarrow relative repres. for x **rnfeltabstorel**(rnf, x)
 relative \rightarrow absolute repres. for x **rnfeltreltoabs**(rnf, x)
 lift x to the relative field **rnfeltup**(rnf, x)
 push x down to the base field **rnfeltdown**(rnf, x)
 idem for x ideal: (**rnfideal**)**reltoabs**, **abstorel**, **up**, **down**

Norms and Trace

relative norm of element $x \in L$ **rnfeltnorm**(rnf, x)
 relative trace of element $x \in L$ **rnfelttrace**(rnf, x)
 absolute norm of ideal x **rnfidealnrmabs**(rnf, x)
 relative norm of ideal x **rnfidealnrmrel**(rnf, x)
 solutions of $N_{K/\mathbf{Q}}(y) = x \in \mathbf{Z}$ **bnfisintnorm**(bnf, x)
 is $x \in \mathbf{Q}$ a norm from K ? **bnfisnorm**($bnf, x, \{flag\}$)
 initialize T for norm eq. solver **rnfisnorminit**($K, pol, \{flag\}$)
 is $a \in K$ a norm from L ? **rnfisnorm**($T, a, \{flag\}$)
 initialize t for Thue equation solver **thueinit**(f)
 solve Thue equation $f(x, y) = a$ **thue**($t, a, \{sol\}$)
 characteristic poly. of a mod T **rnfcharpoly**($nf, T, a, \{v\}$)

Factorization

factor ideal x in L **rnfidealfactor**(rnf, x)
 $[S, T]: T_{i,j} \mid S_i; S$ primes of K above p **rnfidealprimedec**(rnf, p)

Maximal order \mathbf{Z}_L as a \mathbf{Z}_K -module

relative **polredbest** **rnfpolredbest**(nf, T)
 relative Dedekind criterion, prime pr **rnfdedekind**(nf, T, pr)
 discriminant of relative extension **rnfdisc**(nf, T)
 pseudo-basis of \mathbf{Z}_L **rnfpsedobasis**(nf, T)
General \mathbf{Z}_K -modules: $M = [\text{matrix, vec. of ideals}] \subset L$
 relative HNF / SNF **nfhnf**(nf, M), **nfsnf**
 multiple of det M **nfdetint**(nf, M)
 HNF of M where $d = nfdetint(M)$ **nfhnfmod**(x, d)
 reduced basis for M **rnflllgram**(nf, T, M)
 determinant of pseudo-matrix M **rnfdet**(nf, M)
 Steinitz class of M **rnfsteinitz**(nf, M)
 \mathbf{Z}_K -basis of M if \mathbf{Z}_K -free, or 0 **rnfhnfbasis**(bnf, M)
 n -basis of M , or $(n+1)$ -generating set **rnfbasis**(bnf, M)
 is M a free \mathbf{Z}_K -module? **rnfisfree**(bnf, M)

Associative Algebras

A is a general associative algebra given by a mult. table mt (over \mathbf{Q} or \mathbf{F}_p); represented by al from **algtbleinit**.

create al from mt (over \mathbf{F}_p) **algtbleinit**($mt, \{p=0\}$)
 group algebra $\mathbf{Q}[G]$ (or $\mathbf{F}_p[G]$) **alggroup**($G, \{p=0\}$)

Properties

is (mt, p) OK for algtbleinit? **algisassociative**($mt, \{p=0\}$)
 multiplication table mt **algmtable**(al)
 multiplication table over center **algrelmtable**(al)
 dimension of A over prime subfield **algabsdim**(al)
 characteristic of A **algchar**(al)
 is A commutative? **algiscommutative**(al)
 is A simple? **algissimple**(al)
 is A semi-simple? **algissemisimple**(al)
 is A ramified? (at place v) **algisramified**($al, \{v\}$)
 is A split? (at place v) **algissplit**($al, \{v\}$)
 center of A **algcenter**(al)
 Jacobson radical of A **algradical**(al)
 radical J and simple factors of A/J **algdecomposition**(al)
 simple factors of semi-simple A **algsimpledec**(al)

Operations on algebras

create A/I , I two-sided ideal **algquotient**($al, I, \{flag=0\}$)
 create $A_1 \otimes A_2$ **algtensor**($al1, al2$)
 create subalgebra from basis B **algsubalg**(al, B)
 \dots from orthogonal central idempotents e **algcentralproj**(al, e)
 prime subalgebra of semi-simple A over \mathbf{F}_p **algprimesubalg**(al)
 lattice generated by cols. of M **alglathnf**(al, M)

Operations on elements

$a+b, a-b, -a$ **algadd**(al, a, b), **algsub**, **algneg**
 $a \times b, a \times a$ **algmul**(al, a, a), **algsqr**
 a^n, a^{-1} **algpow**(al, a, n), **alginv**
 is x invertible? (then set $z = x^{-1}$) **algisinv**($al, x, \{\&z\}$)
 find z such that $x \times z = y$ **algdivl**(al, x, y)
 find z such that $z \times x = y$ **algdivr**(al, x, y)
 does z s.t. $x \times z = y$ exist? (set it) **algisdivl**($al, x, y, \{\&z\}$)
 matrix of $v \mapsto x \cdot v$ **algleftmtable**(al, x)
 absolute norm **algnorm**(al, x)
 absolute trace **algtrace**(al, x)
 absolute char. polynomial **algcharpoly**(al, x)
 given $a \in A$ and polynomial T , return $T(a)$ **algpoleval**(al, T, a)
 random element in a box **algrandom**(al, b)

Central Simple Algebras

A is a central simple algebra over a number field K ; represented by al from **algnit**; K is given by a nf structure.

create CSA from data **algnit**($B, C, \{v\}, \{flag=0\}$)
 multiplication table over K $B = K, C = mt$
 cyclic algebra $(L/K, \sigma, b)$ $B = rnf, C = [sigma, b]$
 quaternion algebra $(a, b)_K$ $B = K, C = [a, b]$
 matrix algebra $M_d(K)$ $B = K, C = d$
 local Hasse invariants over K $B = K, C = [d, [PR, HF], HI]$

Properties

type of al (mt , CSA) **algtype**(al)
 is al a division algebra? (at place v) **algisdivision**($al, \{v\}$)
 dimension of al over its center **algdim**(al)
 degree of A ($= \sqrt{\dim}$) **algdegree**(al)
 index of A over K (index at v) **algindex**($al, \{v\}$)
 al a cyclic algebra $(L/K, \sigma, b)$; return σ **algaut**(al)
 \dots return b **algb**(al)
 \dots return L/K , as an rnf **algsplittingfield**(al)
 split A over an extension of K **algsplittingdata**(al)
 splitting field of A as an rnf over center **algsplittingfield**(al)
 places of K at which A ramifies **algramifiedplaces**(al)
 Hasse invariants at finite places of K **alghassef**(al)
 Hasse invariants at infinite places of K **alghassei**(al)
 Hasse invariant at place v **alghasse**(al, v)

Operations on elements

reduced norm **algnorm**(al, x)
 reduced trace **algtrace**(al, x)
 reduced char. polynomial **algcharpoly**(al, x)
 express x on integral basis **algalgtobasis**(al, x)
 convert x to algebraic form **algbasistoalg**(al, x)
 map $x \in A$ to $M_d(L)$, L split. field **algsplittingmatrix**(al, x)

Orders

\mathbf{Z} -basis of order \mathcal{O}_0 **algbasis**(al)
 discriminant of order \mathcal{O}_0 **algdisc**(al)
 \mathbf{Z} -basis of natural order in terms \mathcal{O}_0 's basis **alginvbasis**(al)

Based on an earlier version by Joseph H. Silverman

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